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On nonlinear Alfvén-cyclotron waves in multi-species plasma

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Abstract. Large-amplitude Alfvén waves are ubiquitous in space plasmas and a main component of magnetohydrodynamic (MHD) turbulence in the heliosphere. As pump waves, they are prone to parametric instability by which they can generate cyclotron and acoustic daughter waves. Here, we revisit a related process within the framework of the multi-fluid equations for a plasma consisting of many species. The nonlinear coupling of the Alfvén wave to acoustic waves is studied, and a set of compressive and coupled-wave equations for the transverse magnetic field and longitudinal electric field is derived for waves propagating along the mean-field direction. It turns out that slightly compressive Alfvén waves exert, through induced gyro-radius and kinetic-energy modulations, an electromotive force on the particles in association with a longitudinal electric field, which has a potential that is given by the gradient of the transverse kinetic energy of the particles gyrating about the mean field. This in turn drives electric fluctuations (sound and ion-acoustic waves) along the mean magnetic field, which can nonlinearly react back on the transverse magnetic field. Mutually coupled Alfvén-cyclotron–acoustic waves are thus excited, a nonlinear process that can drive a cascade of wave energy in the plasma, and may generate compressive microturbulence. These driven electric fluctuations might have consequences for the dissipation of an MHD turbulence and, thus, for the heating and acceleration of particles in the solar wind.

1. Introduction

Large-amplitude Alfvén waves are ubiquitous in space plasmas, and particularly prominent in the solar wind (Tu and Marsch, 1995; Bruno and Carbone, 2005). They are an essential component of magnetohydrodynamic (MHD) turbulence in the heliosphere and known to originate mainly in the solar coronal holes (Cranmer, 2009). As has been shown in the ample literature, an Alfvén mother (pump) wave is prone to parametric instability (Stenflo, 1976; Derby, 1978; Goldstein, 1978; Longtin and Sonnerup, 1986; Wong and Goldstein, 1986; Brodin and Stenflo, 1988; Viñas and Goldstein, 1991a,b; Hollweg, 1994; Stenflo and Shukla, 2000; Ruderman and Simpson, 2004; Stenflo and Shukla, 2007) by which it can generate cyclotron and acoustic daughter waves that may undergo kinetic effects (Araneda, 1998) and collisionless Landau damping (Inhester, 1990; Araneda et al., 2007). The continuous and wide interest in these waves also comes from their astounding properties, namely that Alfveń-cyclotron waves, like parallel magnetosonic-whistler waves, are nonlinear eigenmodes (Sonnerup and Su, 1967; Stenflo, 1976) of the MHD, and multi-fluid equations as shown below, for propagation along the mean magnetic field.

Nonlinearly excited (Spangler, 1989) acoustic waves appear to be common in space plasmas as well, and density fluctuations (Tu and Marsch, 1995; Bruno and Carbone, 2005) are observed everywhere in the solar wind, although at a comparatively low fluctuation level of merely a few percent. However, since compressive fluctuations can be damped through kinetic effects, like Landau damping on the thermal ions and electrons, they can provide an effective dissipation mechanism for the nonlinear damping (Medvedev et al., 1997) of Alfvén waves. Consequently, the understanding of the coupling between Alfvénic wave activity and density or charge-density fluctuations is of paramount interest and importance in basic plasma physics, but alike in its applications to nonlinear processes in space (Stenflo and Shukla, 2007) and astrophysical plasmas.

As will be shown in this paper, a coupled set of nonlinear second-order wave equations for the transverse magnetic field, the transverse gyromotion of any particle species in the multi-component plasma considered, and the related longitudinal electric field can be derived, which together describe the wave–wave interactions and their mutual forcing. These equations provide a physically and intuitively clear picture of the field and particle/plasma dynamics and allow us to understand the results of recent hybrid simulations of the parametric decay of Alfvén waves and their effects on the plasma particles better. The main aim of this work is to provide algebraic derivations and physical explanations. A numerical treatment of the full equations to be derived subsequently appears promising, yet is beyond the scope of this work.

In analytical (Araneda et al., 2007), hybrid-simulation, and other numerical simulation (Araneda et al., 2008) studies of the parametric instabilities of Alfvén-cyclotron waves, it became obvious that ion trapping (Araneda et al., 2008, 2009) in the nonlinearly driven ion-acoustic waves and pitch-angle scattering by the transverse daughter waves were found to cause anisotropic heating of the proton core velocity distribution, and simultaneously to create a proton beam along the mean field (Araneda et al., 2008; Valentini and Veltri, 2009). These numerical results are in close agreement with observed kinetic features in the solar wind and support the observation that pitch-angle scattering (Marsch and Tu, 2001; Heuer and Marsch, 2007) is the key to understand the kinetic characteristics of thermal solar-wind protons. But only recently convincing evidence has been found for ion-cyclotron waves (Jian et al., 2009) to exist in the solar wind. Also, simulations of electric field spectra (Valentini et al., 2008) have shown that the short-scale termination of solar wind turbulence is characterized by the occurrence of longitudinal electrostatic fluctuations. The spectra thus obtained seem to be consistent with the electrostatic waves actually measured in the solar wind (Bale et al., 2005) close to the Earth's bow shock, and in particular in the ion-cyclotron range (Kellogg et al., 2006).

The present study will provide the foundation for insight into and further study of the processes occurring at macroscopic and microscopic scales in solar-wind turbulence, and thus will throw light on the related dissipation processes through kinetic cascades and wave–particle interactions (Marsch, 2006). The nonlinear equations derived here are used to describe an elliptically polarized Alfvén wave as a simple but nontrivial example of their application.

2. The multi-fluid equations in conservation form

2.1. Fluid equations in the wave frame

In this section, we shall first recapitulate the basic multi-fluid equations (Goossens, 2003) for a plasma consisting of electrons and various ionic species. We start from the fundamental conservation laws, and then try to make no approximations with respect to the field amplitudes in order to be able to discuss and analyze nonlinear waves and convected wave-like structures. We are going to use coordinates in the frame of reference moving with the wave, which has a normal to its front denoted by $\hat{\boldsymbol{n}}$ and a propagation speed $\boldsymbol{V} = V\hat{\boldsymbol{n}}$ in the inertial frame, or centre of momentum frame that is defined below. This unit vector obeys the relation: $\hat{\boldsymbol{n}}^2 = 1$. The coordinate in this moving frame is $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{V}t$, and all variables are assumed to depend on space and time only through $\boldsymbol{\xi}$, and thus spatial and temporal derivatives in the wave frame are reduced to derivatives with respect to $\boldsymbol{\xi}$. Such a coordinate transformation has been used by many authors, for example to study solitary waves in multi-ion plasmas (Hackenberg et al., 1998), or their stability properties (McKenzie et al., 1993). Therefore, by using comoving coordinates, Maxwell's partial differential equations in space and time, and similarly the fluid equations for the different species, can be reduced to simpler differential equations in terms of $\boldsymbol{\xi}$. The continuity equations thus read

$$\frac{\partial}{\partial \boldsymbol{\xi}} \cdot (n_j \boldsymbol{V}_j) = 0, \tag{2.1}$$

with $\boldsymbol{V}_j$ being the flow velocity of species j in the moving frame $\boldsymbol{V}_j = \boldsymbol{U}_j - \boldsymbol{V}$, and n_j is its number density. With the charge denoted as q_j, we get the total charge density as

$$\sigma = \sum_j \sigma_j = \sum_j q_j n_j, \tag{2.2}$$

which must obey Gauss' law.

$$4\pi\sigma = \frac{\partial}{\partial \boldsymbol{\xi}} \boldsymbol{E}. \tag{2.3}$$

Similarly, the total current density is given by

$$\boldsymbol{J} = \sum_j q_j n_j \boldsymbol{U}_j, \tag{2.4}$$

which has to obey Ampère's law:

$$\boldsymbol{J} = \frac{c}{4\pi} \frac{\partial}{\partial \boldsymbol{\xi}} \times \boldsymbol{B} + \frac{1}{4\pi} \frac{\partial}{\partial \boldsymbol{\xi}} (\boldsymbol{V}\boldsymbol{E}). \tag{2.5}$$

The second term of (2.5) is the displacement current. The conduction minus convection current density may be written

$$\boldsymbol{j} = \boldsymbol{J} - \sigma \boldsymbol{V} = \sum_j \sigma_j \boldsymbol{V}_j. \tag{2.6}$$

To complete the set of Maxwell's equations, we quote that the magnetic field must be divergence free

$$\frac{\partial}{\partial \boldsymbol{\xi}} \boldsymbol{B}' = 0, \tag{2.7}$$

and that Faraday's induction equation requires the curl of the electric field in the wave frame to vanish

$$\frac{\partial}{\partial \xi} \times \boldsymbol{E}' = 0, \tag{2.8}$$

with the primed variables being defined in the wave frame. The Lorentz transformation has been used to derive (2.8), since it gives the connection between the electromagnetic fields (c is the speed of light in vacuo) in the plasma's centre of momentum frame and the moving wave frame through the relation.

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{1}{c}\boldsymbol{V} \times \boldsymbol{B}. \tag{2.9}$$

Of course, the magnetic field remains invariant to lowest order in V/c, and thus $\boldsymbol{B}' = \boldsymbol{B}$. For later purposes, we define the mass density of particle kind j as $\varrho_j = n_j m_j$, with the total mass density

$$\varrho = \sum_j \varrho_j. \tag{2.10}$$

The centre of momentum velocity (for which we are free to choose $\boldsymbol{U} = \boldsymbol{0}$) is defined as follows:

$$\varrho \boldsymbol{U} = \sum_j \varrho_j \boldsymbol{U}_j. \tag{2.11}$$

Since we are interested in the individual ion and electron dynamics, we do not sum their momentum equations like in the MHD, but use instead the separate multi-fluid equations. The individual momentum equation of species j can conveniently be quoted in conservation form in the moving frame, reading

$$\frac{\partial}{\partial \xi}\left(m_j n_j \boldsymbol{V}_j \boldsymbol{V}_j + p_j \boldsymbol{1}\right) = q_j n_j \left(\boldsymbol{E}' + \frac{1}{c}\boldsymbol{V}_j \times \boldsymbol{B}'\right). \tag{2.12}$$

The expression $\boldsymbol{V}_j \boldsymbol{V}_j$ means a tensor in dyadic notation, and $\boldsymbol{1}$ is the unit dyade. For the equation of the partial pressure, we may take a simple polytropic equation of state for the purpose of closure, and thus write

$$p_j = p_{j0}\left(\frac{n_j}{n_{j0}}\right)^{\gamma_j}, \tag{2.13}$$

with some constant reference density, n_{j0}, and pressure, p_{j0}. Equivalently, we may consider the (polytropic, with index γ_j) entropy equation

$$\boldsymbol{V}_j \frac{\partial}{\partial \xi} ln\left(p_j \varrho_j^{-\gamma_j}\right) = 0. \tag{2.14}$$

The set of equations (2.1), (2.3), (2.5), (2.7), (2.8), (2.9), (2.12), (2.13) is closed and sufficient to calculate all the independent but coupled variables. In what follows we shall assume a reduced geometry.

2.2. Reduced multi-fluid equations in one-dimensional geometry

We consider a one-dimensional spatial setup, i.e. a dependence on only one spatial coordinate and take components with respect to $\hat{\boldsymbol{n}}$, where the unit vector may correspond to the wave unit vector $\hat{\boldsymbol{k}}$ in Fourier variables. Thus, we generally define the components

$$\xi = \hat{\boldsymbol{n}} \cdot (\boldsymbol{x} - \boldsymbol{V} t), \tag{2.15}$$

$$\boldsymbol{V}_j = V_{j\mathrm{n}} \hat{\boldsymbol{n}} + \boldsymbol{V}_{j\mathrm{t}}, \tag{2.16}$$

$$\boldsymbol{V}_{j\mathrm{t}} = (\boldsymbol{1} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) \cdot \boldsymbol{V}_j. \tag{2.17}$$

Transverse components are obtained by projection perpendicular to the longitudinal direction. The corresponding magnetic field components are defined as $B_\mathrm{n} = \hat{\boldsymbol{n}} \cdot \boldsymbol{B}$ and $\boldsymbol{B}_\mathrm{t} = (\boldsymbol{1} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) \cdot \boldsymbol{B}$, from which it follows that $\hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathrm{t} = \hat{\boldsymbol{n}} \times \boldsymbol{B}$, and that $\boldsymbol{B}_\mathrm{t} \cdot \hat{\boldsymbol{n}} = 0$. The fluid equations then read as follows. For the longitudinal momentum conservation, we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left(n_j V_{j\mathrm{n}} V_{j\mathrm{n}} + \frac{p_j}{m_j} \right) = \frac{q_j n_j}{m_j} \left(E'_\mathrm{n} + \frac{1}{c} \left(\boldsymbol{V}_j \times \boldsymbol{B}' \right) \cdot \hat{\boldsymbol{n}} \right), \tag{2.18}$$

and for transverse momentum conservation, we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left(n_j V_{j\mathrm{n}} \boldsymbol{V}_{j\mathrm{t}} \right) = \frac{q_j n_j}{m_j} \left(\boldsymbol{E}'_\mathrm{t} + \frac{1}{c} \left(\boldsymbol{V}_j \times \boldsymbol{B}' \right)_\mathrm{t} \right). \tag{2.19}$$

The longitudinal magnetic field is strictly conserved and thus constant

$$\frac{\mathrm{d}B'_\mathrm{n}}{\mathrm{d}\xi} = 0, \tag{2.20}$$

and the charge density obeys Gauss' law

$$\frac{\mathrm{d}E'_\mathrm{n}}{\mathrm{d}\xi} = 4\pi \sum_j q_j n_j. \tag{2.21}$$

If we take after (2.8) the curl of $\boldsymbol{E}'$ in the wave frame, we find

$$\hat{\boldsymbol{n}} \frac{\mathrm{d}}{\mathrm{d}\xi} \times \boldsymbol{E}' = \hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{E}'_\mathrm{t}}{\mathrm{d}\xi} = 0 = \hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{E}_\mathrm{t}}{\mathrm{d}\xi} + \hat{\boldsymbol{n}} \times \left(\frac{1}{c} V \hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{B}_\mathrm{t}}{\mathrm{d}\xi} \right), \tag{2.22}$$

where the last part of the equation follows from the previous (2.9). In conclusion, $\boldsymbol{E}'_\mathrm{t}$ is constant and can be set equal to zero. Using this result, the transverse electric field is obtained through (2.9), which is equivalent to writing

$$\boldsymbol{E}_\mathrm{t} = -\frac{1}{c} V (\hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathrm{t}), \tag{2.23}$$

and makes the transverse electric field a dependent auxiliary variable being fully determined by the transverse magnetic field. For the longitudinal electric field component in the wave frame one has

$$E'_\mathrm{n} = E_\mathrm{n} + \frac{1}{c} \hat{\boldsymbol{n}} \cdot (V \hat{\boldsymbol{n}} \times \boldsymbol{B}), \tag{2.24}$$

which yields $E'_\mathrm{n} = E_\mathrm{n}$, which is to be used in Gauß' law (2.21). For the one-dimensional spatial geometry chosen here, the longitudinal current density can be written as

$$J_\mathrm{n} = \frac{c}{4\pi} \hat{\boldsymbol{n}} \cdot \left(\hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}\xi} \right) + \frac{1}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\xi} (V E_\mathrm{n}). \tag{2.25}$$

Since the curl of $\boldsymbol{B}$ has only transverse components, we obtain from (2.25) that the longitudinal current density in the wave frame must be strictly constant, since it is given by

$$j_{\mathrm{n}} = \sum_j q_j n_j V_{j\mathrm{n}} = \sum_j \sigma_j V_{j\mathrm{n}} = \sum_j q_j F_{j\mathrm{n}}, \tag{2.26}$$

where the individual particles fluxes ($F_{j\mathrm{n}}$) are conserved according to the longitudinal continuity equation, which expresses flux conservation in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(n_j V_{j\mathrm{n}}\right) = \frac{\mathrm{d}F_{j\mathrm{n}}}{\mathrm{d}\xi} = 0. \tag{2.27}$$

The transverse component of Ampère's law including the induction current can be cast in the form

$$\boldsymbol{j}_{\mathrm{t}} = \frac{c}{4\pi}\left(\hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{B}_{\mathrm{t}}}{\mathrm{d}\xi}\right) + \frac{1}{4\pi} V \frac{\mathrm{d}\boldsymbol{E}_{\mathrm{t}}}{\mathrm{d}\xi} = \sum_j q_j n_j \boldsymbol{V}_{j\mathrm{t}}, \tag{2.28}$$

whereby we note that $\boldsymbol{V}_{j\mathrm{t}} = \boldsymbol{U}_{j\mathrm{t}}$, since $\boldsymbol{V}$ has no transverse component, but $\boldsymbol{V}_{j\mathrm{n}} = \boldsymbol{U}_{j\mathrm{n}} - V$. The right-hand side of (2.22) gives an expression for the gradient of the transverse electric field component. It can be inserted in Ampère's law, which thus can be written as

$$\frac{1}{4\pi} V \frac{\mathrm{d}\boldsymbol{E}_{\mathrm{t}}}{\mathrm{d}\xi} = -\frac{1}{4\pi c} V^2 \left(\hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{B}_{\mathrm{t}}}{\mathrm{d}\xi}\right) = \sum_j n_j q_j \boldsymbol{V}_{j\mathrm{t}} - \frac{c}{4\pi}\left(\hat{\boldsymbol{n}} \times \frac{\mathrm{d}\boldsymbol{B}_{\mathrm{t}}}{\mathrm{d}\xi}\right). \tag{2.29}$$

If $(V/c)^2 \ll 1$, which we will assume in the remainder, then the displacement current term can be safely neglected. It is certainly needed if one wants to make the transition to free electromagnetic waves correctly, in which we are not interested here. We will therefore not keep this term anymore. The basic equation for the magnetic field, which is Ampère's law for the transverse component, then reads

$$\frac{4\pi}{c} \sum_j n_j q_j \boldsymbol{V}_{j\mathrm{t}} = \frac{\mathrm{d}}{\mathrm{d}\xi}(\hat{\boldsymbol{n}} \times \boldsymbol{B}_{\mathrm{t}}). \tag{2.30}$$

The magnetic field is free of divergence, which means in our geometry and variables that (2.20) is fulfilled with $B_{\mathrm{n}} = B'_{\mathrm{n}}$. The last two equations together fully determine the vector magnetic field, given the current density is provided. The transverse electric field is obtained from (2.23) and the longitudinal one from Gauß' law (2.21) with $E_{\mathrm{n}} = E'_{\mathrm{n}}$.

Next, we quote again the transverse momentum equation for each species, which reads

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(n_j V_{j\mathrm{n}} \boldsymbol{V}_{j\mathrm{t}}\right) = \frac{q_j}{m_j c} n_j \left(\boldsymbol{V}_j \times \boldsymbol{B}\right)_{\mathrm{t}} = \frac{q_j n_j}{m_j c}\left(V_{j\mathrm{n}} \hat{\boldsymbol{n}} \times \boldsymbol{B}_{\mathrm{t}} + \boldsymbol{V}_{j\mathrm{t}} \times \hat{\boldsymbol{n}} B_{\mathrm{n}}\right), \tag{2.31}$$

whereby the electric field term has been written out in detail. Similarly, the longitudinal momentum equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(n_j V_{j\mathrm{n}}^2 + \frac{p_j}{m_j}\right) = \frac{q_j n_j}{m_j}\left(E_{\mathrm{n}} + \frac{1}{c}(\boldsymbol{V}_j \times \boldsymbol{B}) \cdot \hat{\boldsymbol{n}}\right), \tag{2.32}$$

which must be supplemented to obtain closure by the entropy or pressure equation,

$$\frac{\mathrm{d}}{\mathrm{d}\xi} ln\left(p_j \varrho_j^{-\gamma_j}\right) = 0. \tag{2.33}$$

In what follows it turns out to be convenient to use the natural spatial and temporal scales of the multicomponent plasma, which depend on the various fluid and field parameters. We define a longitudinal gyration length for the species j as

$$r_j = \frac{V_{jn}}{\Omega_j} = \frac{F_{jn}}{\Omega_j}\,\frac{1}{n_j}, \tag{2.34}$$

which corresponds to the gyration radius calculated with the longitudinal velocity instead of the perpendicular one. It is implicitly dependent on n_j via the drift speed and (2.27). Another interesting length is given by the strictly constant quantity

$$L_j = \frac{B_n c}{4\pi q_j F_{jn}} = r_j \left(\frac{V_{Aj}}{V_{jn}}\right)^2 = \frac{1}{r_j}\left(\frac{c}{\omega_j}\right)^2. \tag{2.35}$$

Here, we have introduced the Alfvén speed, $V_{Aj}^2 = B_n^2/(4\pi n_j m_j)$, based on the mass density ϱ_j of species j only, and the respective plasma frequency $\omega_j^2 = 4\pi q_j^2 n_j/m_j$ and gyrofrequency $\Omega_j = B_n q_j/(m_j c)$, carrying the sign of the charge q_j. Note that L_j is strictly constant, and its inverse sums over all species up to zero, i.e. $\sum_j 1/L_j = 0$, because of the condition of zero longitudinal total current: $j_n - \sigma V = \sum_j q_j n_j V_{jn} = 0$. The standard Alfvén speed based on B_n is obtained by the summation

$$\sum_j \frac{1}{V_{Aj}^2} = \frac{1}{V_A^2}. \tag{2.36}$$

Concerning the compressive dynamics, it is important to note that longitudinal and transverse motions are coupled through p_j and $r_j = V_{jn}/\Omega_j$, i.e. through the particle number density, when the mass continuity (2.27) and entropy (2.33) equations are exploited.

3. Wave equations

In this section, we recast the basic equations for the fields and plasma multi-fluid parameters into the form of coupled wave equations. For that purpose, we shall not straightforwardly Fourier transform them but rather rewrite them, by use of multiple differentiation, in such a form that we finally obtain single “wave equations” for the electric field and magnetic field components. Remember that in the moving frame all variables depend solely on the coordinate $\xi = \boldsymbol{x}\cdot\hat{\boldsymbol{n}} - Vt$. We first consider the pressure equation again. We may use for the species' sound speed the standard definition.

$$c_j^2 = \frac{\partial p_j}{\partial \varrho_j}, \tag{3.1}$$

and can then re-evaluate the momentum equation by use of

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{p_j}{m_j}\right) = c_j^2 \frac{\mathrm{d}n_j}{\mathrm{d}\xi}, \tag{3.2}$$

which together with (2.27) allows us to quote the longitudinal momentum equation in the form

$$\left(c_j^2 - V_{jn}^2\right)\frac{\mathrm{d}n_j}{\mathrm{d}\xi} = \frac{q_j n_j}{m_j}\left(E_n + \frac{1}{c}(\boldsymbol{V}_{jt}\times\boldsymbol{B}_t)\cdot\hat{\boldsymbol{n}}\right). \tag{3.3}$$

It is convenient to introduce the effective Debye length λ_j of species j as follows:

$$\frac{1}{\lambda_j^2} = \frac{\omega_j^2}{c_j^2 - V_{j\mathrm{n}}^2}, \tag{3.4}$$

the sum of which gives the total Debye length, still including the differential drifts

$$\frac{1}{\lambda_\mathrm{D}^2} = \sum_j \frac{1}{\lambda_j^2}. \tag{3.5}$$

Both λ_j and λ_D are not necessarily real quantities. Each species brings in its own length scale λ_j. It is also convenient to introduce the second-order wave operator (which parametrically depends still on V via the $V_{j\mathrm{n}}$).

$$\mathscr{D}_E = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \frac{1}{\lambda_\mathrm{D}^2}. \tag{3.6}$$

Finally, one obtains a driven wave equation for the longitudinal electric field

$$\mathscr{D}_E\, E_\mathrm{n} = \frac{1}{c}\sum_j \frac{1}{\lambda_j^2}(\boldsymbol{V}_{j\mathrm{t}} \times \boldsymbol{B}_\mathrm{t}) \cdot \hat{\boldsymbol{n}}, \tag{3.7}$$

in which the transverse particle motions and electromagnetic fields show up through a nonlinear electromotive force, which is the summed contribution of the longitudinal components of the Lorentz forces acting on each species. This driving force acting on E_n resembles a "convection" electric field. When being decoupled from the transverse plasma and field dynamics, the longitudinal electric field equation just describes free electrostatic oscillations, such as Langmuir and acoustic waves, as we will show later.

Let us return to the transverse momentum equation and rewrite it by exploiting the mass continuity equation. Then it is straightforward to derive

$$\frac{\mathrm{d}\boldsymbol{V}_{j\mathrm{t}}}{\mathrm{d}\xi} = \frac{1}{r_j}\left(\frac{V_{j\mathrm{n}}}{B_\mathrm{n}}\,\hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathrm{t} - \hat{\boldsymbol{n}} \times \boldsymbol{V}_{j\mathrm{t}}\right). \tag{3.8}$$

Using this equation, we can rewrite the normal component of the convection electric field that occurs in (3.3) and (3.7) as follows:

$$E_{j\mathrm{n}} \equiv \frac{1}{c}\left(\boldsymbol{V}_{j\mathrm{t}} \times \boldsymbol{B}_\mathrm{t}\right) \cdot \hat{\boldsymbol{n}} = -\frac{B_\mathrm{n}}{V_{j\mathrm{n}}}\frac{r_j}{c}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\boldsymbol{V}_{j\mathrm{t}}^2\right) = -\frac{m_j}{q_j}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\boldsymbol{V}_{j\mathrm{t}}^2\right), \tag{3.9}$$

a relation, which is going to be used later. If the module of the transverse plasma velocity of species j is constant then its convection electric field $E_{j\mathrm{n}}$ vanishes. Using Ampère's law (2.30) and the previous (3.9), we can derive by vector cross multiplication of (2.30) with $\hat{\boldsymbol{n}}$ and subsequent scalar multiplication with $\boldsymbol{B}_\mathrm{t}$ another conservation law.

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_j \varrho_j \boldsymbol{V}_{j\mathrm{t}}^2 - \frac{1}{4\pi}\boldsymbol{B}_\mathrm{t}^2\right) = 0. \tag{3.10}$$

If the integration constant is zero, this equation expresses equipartition between the transverse total particle kinetic energy and the transverse magnetic energy, like it is the case in a classical MHD Alfvén wave.

We consider now the equation for the transverse magnetic field. By differentiation of Ampère's law, we can obtain a second-order nonlinear wave equation for the transverse magnetic field

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\,\hat{\boldsymbol{n}} \times \boldsymbol{B}_{\mathrm{t}} = \frac{4\pi}{c}\sum_j q_j\left(n_j\frac{\mathrm{d}\boldsymbol{V}_{j\mathrm{t}}}{\mathrm{d}\xi} + \boldsymbol{V}_{j\mathrm{t}}\frac{\mathrm{d}n_j}{\mathrm{d}\xi}\right). \tag{3.11}$$

Here, it is convenient to introduce the skin depth or inertial length ℓ_j of species j defined as follows:

$$\frac{1}{\ell_j^2} = \frac{\omega_j^2}{c^2}, \tag{3.12}$$

the sum of which gives the total skin depth:

$$\frac{1}{\ell_{\mathrm{S}}^2} = \sum_j \frac{1}{\ell_j^2}, \tag{3.13}$$

where each species brings in its own length scale ℓ_j. It is again convenient to introduce a second-order wave operator

$$\mathscr{D}_B = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \frac{1}{\ell_{\mathrm{S}}^2}. \tag{3.14}$$

Using this, one finally obtains a driven wave equation for the transverse magnetic field

$$\mathscr{D}_B\,\boldsymbol{B}_{\mathrm{t}} = \sum_j\left(-\frac{B_{\mathrm{n}}}{V_{j\mathrm{n}}}\frac{1}{\ell_j^2}\boldsymbol{V}_{j\mathrm{t}} + \frac{1}{\lambda_j^2\,c}(\hat{\boldsymbol{n}} \times \boldsymbol{V}_{j\mathrm{t}})\left[E_{\mathrm{n}} + \frac{1}{c}(\boldsymbol{V}_{j\mathrm{t}} \times \boldsymbol{B}_{\mathrm{t}})\cdot\hat{\boldsymbol{n}}\right]\right). \tag{3.15}$$

On the right-hand side, the transverse currents appear and the longitudinal charge density variations and related electrostatic effects show up through the nonlinear electromotive force, which involves the longitudinal electric field. Note that this nonlinear driver contains the natural length scales (density dependent) of all the species involved. Of course, this wave equation seems, without further approximation, quite formal but elucidates the nature of the coupling of the unforced transverse magnetic field (dynamics described by the operator $\mathscr{D}_B$) with the compressive electrostatic fluctuations and transverse plasma motions. When being decoupled from the plasma currents and electric-field (no charges) dynamics, this transverse magnetic field equation just describes the finite penetration of the magnetic field into the skin layer of the plasma and results in its exponential decline on the length scale ℓ_{S}. To gain better insight into the terms contributing to (3.15), we may rewrite it also in the form

$$\mathscr{D}_B\,\boldsymbol{B}_{\mathrm{t}} = \sum_j \frac{\omega_j^2}{c^2}\left(-\frac{B_{\mathrm{n}}}{V_{j\mathrm{n}}}\boldsymbol{V}_{j\mathrm{t}} + \frac{c(E_{\mathrm{n}} + E_{j\mathrm{n}})}{c_j^2 - V_{j\mathrm{n}}^2}(\hat{\boldsymbol{n}} \times \boldsymbol{V}_{j\mathrm{t}})\right). \tag{3.16}$$

Similarly, one can also rewrite the driven wave equation for the longitudinal electric field in the concise form

$$\mathscr{D}_E\,E_{\mathrm{n}} = \sum_j \frac{1}{\lambda_j^2}E_{j\mathrm{n}}, \tag{3.17}$$

where we remind that $E_{j\mathrm{n}}$ can be derived after (3.9) from a potential that is given by the transverse kinetic energy of species j. These two coupled nonlinear equations are completed and closed by the equation (3.3) for the density and (3.8) for the transverse velocity of each species.

So far, we made neither any approximation nor linearization, but just inserted the original momentum equation (3.3) and (3.8) into the differentiated Ampère's and Gauß' laws. Therefore, the above equations depend on a highly nonlinear manner still on the different number densities n_j. Yet the transition in (3.16) and (3.8) to the incompressible limit is simple, because then the electrostatic nonlinear forcing terms E_{n} and $E_{j\mathrm{n}}$ vanish, and the plasma frequency, ω_j, respectively, the gyration scale r_j, become constants as defined by the fixed background density of species j. We discuss this in the section after the next one.

4. Eigenmodes and driven waves

4.1. Nonlinear Alfvén-ion-cyclotron waves

In order to maintain the linear form of the original equations, it is convenient to introduce new variables, relating to left- and right-hand polarized fields, which are defined as follows:

$$\boldsymbol{B}_{\mathrm{t}}^{\pm} = \boldsymbol{B}_{\mathrm{t}} \pm (\hat{\boldsymbol{n}} \times \boldsymbol{B}_{\mathrm{t}}), \tag{4.1}$$

$$\boldsymbol{V}_{\mathrm{t}}^{\pm} = \boldsymbol{V}_{\mathrm{t}} \pm (\hat{\boldsymbol{n}} \times \boldsymbol{V}_{\mathrm{t}}). \tag{4.2}$$

These field variables are orthogonal, i.e. $\boldsymbol{B}_{\mathrm{t}}^{\pm} \cdot \boldsymbol{B}_{\mathrm{t}}^{\mp} = 0$, and $\boldsymbol{V}_{j\mathrm{t}}^{\pm} \cdot \boldsymbol{V}_{j\mathrm{t}}^{\mp} = 0$. By taking the cross product of (2.30) and (3.8) with the unit vector $\hat{\boldsymbol{n}}$, we obtain after some algebra the equations of motion for the circular transverse variables.

$$\frac{\mathrm{d}\boldsymbol{B}_{\mathrm{t}}^{\pm}}{\mathrm{d}\xi} = \pm\frac{4\pi}{c}\sum_j \sigma_j \boldsymbol{V}_{j\mathrm{t}}^{\mp}, \tag{4.3}$$

$$\frac{\mathrm{d}\boldsymbol{V}_{j\mathrm{t}}^{\pm}}{\mathrm{d}\xi} = \mp\frac{1}{r_j}\left(\frac{V_{j\mathrm{n}}}{B_{\mathrm{n}}}\boldsymbol{B}_{\mathrm{t}}^{\mp} - \boldsymbol{V}_{j\mathrm{t}}^{\mp}\right). \tag{4.4}$$

Let us now first consider nonlinear incompressible solutions. Since $E_{\mathrm{n}} = 0$, consequently, quasi-neutrality strictly holds, $\sum_j q_j n_j = 0$. The velocity fields and magnetic field must have constant modules and be aligned, so that after (3.9) their respective vector cross product, and thus also $E_{j\mathrm{n}}$ vanishes. As n_j, r_j, $V_{j\mathrm{n}}$, c_j and ω_j then all are constant, we can solve the resulting linear set (4.3) and (4.4) by Fourier transformation (FT), without putting any limitations on the amplitudes of $\boldsymbol{B}_{\mathrm{t}}^{\pm}$ or $\boldsymbol{V}_{j\mathrm{t}}^{\pm}$, other than (3.10) which implies that the magnetic field amplitude is also constant. As usually, FT means that $\mathrm{d}/\mathrm{d}\xi \to ik$, and thus we can invert the transverse momentum equation (3.8), which yields with the normalized wave vector, $\kappa_j = kV_{j\mathrm{n}}/\Omega_j$, the complex vector relation

$$\tilde{\boldsymbol{V}}_{j\mathrm{t}}(k) = \frac{V_{j\mathrm{n}}}{B_{\mathrm{n}}(1-\kappa_j^2)}\left(\tilde{\boldsymbol{B}}_{\mathrm{t}}(k) + i\kappa_j\,\hat{\boldsymbol{n}} \times \tilde{\boldsymbol{B}}_{\mathrm{t}}(k)\right). \tag{4.5}$$

This result can be inserted into the FT of (3.16) to obtain the algebraic wave equation.

$$\left(k^2 + \frac{1}{\ell_{\mathrm{S}}^2}\right)\tilde{\boldsymbol{B}}_{\mathrm{t}}(k) = \sum_j \frac{1}{\ell_j^2}\frac{1}{1-\kappa_j^2}\left(\tilde{\boldsymbol{B}}_{\mathrm{t}}(k) + i\kappa_j(\hat{\boldsymbol{n}} \times \tilde{\boldsymbol{B}}_{\mathrm{t}}(k))\right), \tag{4.6}$$

which may also be written as

$$\left(k^2 - \sum_j \frac{1}{\ell_j^2}\frac{\kappa_j^2}{1-\kappa_j^2}\right)\tilde{\boldsymbol{B}}_{\rm t}(k) = i\left(\sum_j \frac{1}{\ell_j^2}\frac{\kappa_j}{1-\kappa_j^2}\right)(\hat{\boldsymbol{n}}\times\tilde{\boldsymbol{B}}_{\rm t}(k)), \tag{4.7}$$

and which yields, by taking the vector cross product of (4.7) with $\hat{\boldsymbol{n}}$ and by resolving the resulting two equations, the two dispersion relations describing left- and right-hand polarized waves as follows:

$$k^2 = \sum_j \left(\frac{\omega_j}{c}\right)^2 \frac{\pm\kappa_j}{1\mp\kappa_j} = \sum_j \hat{\varrho}_j \left(\frac{\Omega_j}{V_{\rm A}}\right)^2 \frac{\pm\kappa_j}{1\mp\kappa_j}, \tag{4.8}$$

with the fractional mass density $\hat{\varrho}_j = \varrho_j/\varrho$. Equation (4.8) is nothing else but the standard dispersion relation for the Alfvén-ion-cyclotron (and magnetosonic-whistler) waves in a multi-component plasma with the differential drifts contained in $V_{j{\rm n}}$ and for parallel propagation (see, e.g. Davidson, 1983), yet which applies here to arbitrarily large wave amplitudes. Apparently, the wave frequency is obtained by the Doppler shift formula, $\omega = kV$, where V is hidden in $V_{j{\rm n}} = U_{j{\rm n}} - V$, i.e. in $\kappa_j = kV_{j{\rm n}}/\Omega_j$. Once the wave vector $k = k(V)$ is known, the wave frequency is obtained as a function of the phase speed V.

Let us consider the long-wavelength limit of (4.8), which when being expanded to second order in κ_j reads.

$$1 = \sum_j \hat{\varrho}_j \left(\frac{\Omega_j}{kV_{\rm A}}\right)^2 (\pm\kappa_j)(1\pm\kappa_j), \tag{4.9}$$

where the first term of the sum vanishes, since $\sum_j \left(\frac{\omega_j^2}{\Omega_j}\right) V_{j{\rm n}} = 0$ because of the quasi-neutrality condition, i.e. $\sigma = 0$ in (2.2), and the zero-longitudinal-current constraint (2.26). The second term then yields

$$1 = \sum_j \hat{\varrho}_j \frac{(U_{j{\rm n}}^2 - 2U_{j{\rm n}}V + V^2)}{V_{\rm A}^2}. \tag{4.10}$$

As a vanishing bulk speed, $\boldsymbol{U} = \boldsymbol{0}$, may be assumed, the centre-of-momentum condition means that $\sum_j \hat{\varrho}_j U_{j{\rm n}} = 0$, and thus we can solve for the phase speed in the centre of momentum frame and obtain

$$V = \pm V_{\rm A}\sqrt{1 - \sum_j \hat{\varrho}_j \left(\frac{U_{j{\rm n}}}{V_{\rm A}}\right)^2}. \tag{4.11}$$

This is the phase speed of an Alfvén wave in a multi-component plasma, including field-aligned drift motions, leading to a slowing down of the phase speed.

We shall now derive the incompressible Alfvén-cyclotron wave without resort to the FT, but recourse instead on (3.9). Since $E_{j{\rm n}} = 0$, each velocity vector and magnetic field must be aligned, which generally implies that $\boldsymbol{V}_{j{\rm t}} = a_j\boldsymbol{B}_{\rm t}$. This can be inserted in (3.8) to obtain

$$\frac{{\rm d}\boldsymbol{V}_{j{\rm t}}}{{\rm d}\xi} = \frac{1}{r_j}\left(\frac{V_{j{\rm n}}}{B_{\rm n}a_j} - 1\right)\hat{\boldsymbol{n}}\times\boldsymbol{V}_{j{\rm t}}. \tag{4.12}$$

Twofold differentiation yields the simple harmonic oscillator equation for the gyromotion

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+k_j^2\right)\boldsymbol{V}_{j\mathrm{t}}=\mathbf{0}, \tag{4.13}$$

with the squared wave vector defined as

$$k_j^2=\frac{1}{r_j^2}\left(\frac{V_{j\mathrm{n}}}{B_\mathrm{n}a_j}-1\right)^2. \tag{4.14}$$

Since all species must spatially oscillate in the same way, the wave vector must not depend on the index j, i.e. we can put $k_j=\pm k$, which yields with $\kappa_j=kr_j$ two possible solutions for the desired proportionality coefficient:

$$a_j=\frac{V_{j\mathrm{n}}}{B_\mathrm{n}}\frac{1}{1\pm\kappa_j}. \tag{4.15}$$

Knowing the coefficient a_j, we can use it in the wave (3.16) without electric fields and get another harmonic oscillator equation.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+q^2\right)\boldsymbol{B}_\mathrm{t}=\mathbf{0}, \tag{4.16}$$

where the squared wave vector q is an abbreviation for exactly the same sum as appearing on the right-hand side of (4.8). Since all velocities and the magnetic field are aligned, the wave vector q must be equal to k, and thus we again obtain the same dispersion relation as in the previous Fourier analysis. Finally, the polarization relation (with the plus sign for incompressible Alfvén-cyclotron and minus for magnetosonic-whistler waves) reads

$$\boldsymbol{V}_{j\mathrm{t}}=\frac{V_{j\mathrm{n}}}{B_\mathrm{n}}\frac{1}{1\pm\kappa_j}\boldsymbol{B}_\mathrm{t}. \tag{4.17}$$

With this result, we can further evaluate the conservation law (3.10), and after some algebra obtain the result that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_j\hat{\varrho}_jV_{j\mathrm{n}}^2\frac{1}{(1\pm\kappa_j)^2}-V_\mathrm{A}^2\right)=0. \tag{4.18}$$

Note that the dispersion relation (4.8) can also be cast into the form

$$V_\mathrm{A}^2=\sum_j\hat{\varrho}_jV_{j\mathrm{n}}^2\frac{\mp\kappa_j}{\kappa_j^2(1\pm\kappa_j)}, \tag{4.19}$$

which facilitates a comparison with the previous equation. Expansion of (4.18) and (4.19) to lowest order in $\kappa_j=kr_j$ yields the MHD dispersion relation (4.10), i.e. in this case, the conservation (4.18) has a zero integration constant and simply expresses equipartition between kinetic and magnetic energy densities. This is not true any more if finite gyrokinetic effects are considered.

4.2. Linear electrostatic waves

Let us now discuss pure linear electrostatic waves, which are obtained by taking the trivial solutions, $\boldsymbol{V}_{j\mathrm{t}}=\boldsymbol{B}_\mathrm{t}=\mathbf{0}$, of the wave (3.16), which also implies that $E_{j\mathrm{n}}=0$.

Then the linearized electrostatic wave (3.17) simply reads $\mathscr{D}_E E_{\mathrm{n}} = 0$. After FT, we obtain that $(k^2 + \lambda_{\mathrm{D}}^{-2})\tilde{E}_{\mathrm{n}}(k) = 0$, which explicitly yields the dispersion relation

$$k^2 = \sum_j \frac{\omega_j^2}{(U_{j\mathrm{n}} - V)^2 - c_j^2}. \tag{4.20}$$

We may only consider here the case of zero drifts, i.e. $U_{j\mathrm{n}} = 0$, and a simple electron-proton plasma. Then, we always find two solutions for V^2 from the equation

$$k^2 = \frac{\omega_{\mathrm{e}}^2}{V^2 - c_{\mathrm{e}}^2} + \frac{\omega_{\mathrm{p}}^2}{V^2 - c_{\mathrm{p}}^2}. \tag{4.21}$$

In the long-wavelength limit ($k \to 0$), the diverging phase speed $V_{\mathrm{L}}(k) = \omega_{\mathrm{P}}/k$ corresponds to the Langmuir wave, with the total plasma frequency being defined by $\omega_{\mathrm{P}}^2 = \omega_{\mathrm{e}}^2 + \omega_{\mathrm{p}}^2$. For the ion-acoustic or sound wave, we obtain the constant speed defined as

$$V_{\mathrm{S}} = \sqrt{\frac{\omega_{\mathrm{e}}^2 c_{\mathrm{p}}^2 + \omega_{\mathrm{p}}^2 c_{\mathrm{e}}^2}{\omega_{\mathrm{P}}^2}} = \sqrt{\frac{k_{\mathrm{B}}(\gamma_{\mathrm{e}} T_{\mathrm{e}} + \gamma_{\mathrm{p}} T_{\mathrm{p}})}{m_{\mathrm{e}} + m_{\mathrm{p}}}}. \tag{4.22}$$

Here, k_{B} is Boltzmann's constant. The general solution in terms of frequency follows from the biquadratic equation:

$$\omega^4 - \omega^2\left(\omega_{\mathrm{e}}^2 + \omega_{\mathrm{p}}^2 + (c_{\mathrm{e}}k)^2 + (c_{\mathrm{p}}k)^2\right) + (c_{\mathrm{e}}k)^2(c_{\mathrm{p}}k)^2 + \omega_{\mathrm{e}}^2(c_{\mathrm{p}}k)^2 + \omega_{\mathrm{p}}^2(c_{\mathrm{e}}k)^2 = 0. \tag{4.23}$$

In the short-wavelength limit ($k \to \infty$), we obtain two solutions corresponding to the proton-acoustic wave, with $\omega \approx kc_{\mathrm{p}}$, or electron-acoustic wave with, $\omega \approx kc_{\mathrm{e}}$. Both modes are usually strongly Landau damped if a thermal Vlasov description of the plasma is used. The sound wave, however, can exist since $m_{\mathrm{e}} \ll m_{\mathrm{p}}$, and thus $V_{\mathrm{S}} \approx \sqrt{k_{\mathrm{B}}\gamma_{\mathrm{e}} T_{\mathrm{e}}/m_{\mathrm{p}}}$ for $T_{\mathrm{e}} > T_{\mathrm{p}}$, so that strong proton Landau damping can be avoided.

For a multicomponent plasma with drifts the structure of the eigenmodes becomes correspondingly richer, as each species contributes its own plasma frequency and thermal speed, as well as drift speed. In the presence of a compressive transverse wave, these modes all become coupled and are driven by the nonlinear ponderomotive electric fields $E_{j\mathrm{n}}$ according to (3.17). Similarly, the transverse eigenmodes defined by (4.8) are driven according to (3.16) by the longitudinal electric field E_{n} and the combined action of the various $E_{j\mathrm{n}}$.

4.3. *Compressive Alfvén-cyclotron–acoustic waves*

In this section, we will consider the coupling between the nonlinear electromagnetic Alfvén-cyclotron waves and the electrostatic modes. We recall that no assumptions, such as incompressibility, had to be made as to derive the wave equations (3.16) and (3.17). We shall also rewrite the transverse momentum equation (3.8), which describes the gyromotion as a second-order wave equation. Since the longitudinal gyration scale r_j depends on the density according to (2.34), its differentiation has to be considered. If we neglect the magnetic field for a moment, then we get for the transverse motion an equation in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{1}{r_j^2} - \frac{\mathrm{d}\ln n_j}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\boldsymbol{V}_{j\mathrm{t}} = 0. \tag{4.24}$$

Mathematically speaking, this is the well-known equation for a harmonic oscillator, with an amplitude that may vary exponentially in ξ at a scale set by the density gradient length. For the differential operator yielding harmonic oscillations (first two terms of (4.24)), we introduce the symbol $\mathscr{D}_{V_j}$ to be used below. For the density gradient term, we may approximately write

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln n_j = \frac{\mathrm{d}}{\mathrm{d}\xi}\ln(\bar{n}_j + \delta n_j) \approx \frac{1}{\bar{n}_j}\frac{\mathrm{d}}{\mathrm{d}\xi}\delta n_j = \frac{\delta n_j}{\bar{n}_j}\frac{\mathrm{d}\ln\delta n_j}{\mathrm{d}\xi} \tag{4.25}$$

with the average constant background density $\bar{n}_j$. If the gradient is positive (negative), and therefore the density increases (decreases), the longitudinal scale and thus the amplitude of $\boldsymbol{V}_{jt}$ will decrease (increase) correspondingly. However, as long as the relative density variation remains small, a few percent say, this change will occur on a much larger scale than r_j, namely $\bar{L}_j = r_j\bar{n}_j/\delta n_j$. If the density fluctuates about zero, the net effect of the density modulation on $\boldsymbol{V}_{jt}$ will remain comparatively small.

For the sake of consistency, we will, in the remainder of this section, fully retain this density-induced possible amplitude variation of $\boldsymbol{V}_{jt}$ according to (4.24), but later on neglect the density variations and consider all density-dependent parameters to be fixed at their background values without denoting them explicitly by a barred symbol. Yet, remember that the essential and lowest order variations of the densities of all species have been considered and already taken care of in Gauß' law and the dynamics of E_n, which indeed is of order unity as the background electric field is zero, and similarly in Ampère's law through the appearance of the electric fields E_n and $E_{j\mathrm{n}}$.

We introduce, by using conserved or constant quantities, normalized variables, such that $\boldsymbol{v}_{jt} = \boldsymbol{V}_{jt}/V$, and $\boldsymbol{b}_\mathrm{t} = \boldsymbol{B}_\mathrm{t}/B_\mathrm{n}$. Similarly, we introduce normalized electric fields as follows:

$$e_{j\mathrm{n}} = \frac{cE_{j\mathrm{n}}}{B_\mathrm{n}V}, \quad e_\mathrm{n} = \frac{cE_\mathrm{n}}{B_\mathrm{n}V}. \tag{4.26}$$

We recall (3.16), according to which $e_{j\mathrm{n}}$ is just an abbreviation for the gradient of a potential given by the transverse kinetic energy, for which we have in normalized form:

$$e_{j\mathrm{n}} = -\frac{V}{\Omega_j}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{2}\boldsymbol{v}_{jt}^2\right). \tag{4.27}$$

Using the same normalization for the transverse electric field (2.23), we obtain simply that

$$\boldsymbol{e}_\mathrm{t} = -(\hat{\boldsymbol{n}} \times \boldsymbol{b}_\mathrm{t}), \tag{4.28}$$

which is fully determined by the solution for $\boldsymbol{b}_\mathrm{t}$. Like $\boldsymbol{e}_\mathrm{t}$, the charge densities σ_j are now merely auxiliary quantities and obtained from an integration of the previous (3.3), which in the new variables can be written as

$$r_j\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\sigma_j = (e_\mathrm{n} + e_{j\mathrm{n}})\frac{VV_{j\mathrm{n}}}{c_j^2 - V_{j\mathrm{n}}^2}, \tag{4.29}$$

and then be formally integrated with the result

$$\sigma_j(\xi) = q_j\bar{n}_j\exp\left(V\Omega_j\int_{\bar{\xi}}^{\xi}\mathrm{d}\xi'\,\frac{e_\mathrm{n}(\xi') + e_{j\mathrm{n}}(\xi')}{c_j^2(\xi') - V_{j\mathrm{n}}^2(\xi')}\right). \tag{4.30}$$

As an outcome of all the above considerations, we can now summarize the resulting set of fluid wave equations. Firstly, we obtain for each species' transverse motion a forced and amplitude-modulated harmonic oscillator equation reading

$$\mathscr{D}_{V_j}\, \boldsymbol{v}_{j\mathrm{t}} = \left(\frac{V^2}{c_j^2 - V_{j\mathrm{n}}^2} (e_\mathrm{n} + e_{j\mathrm{n}}) \frac{\mathrm{d}\boldsymbol{v}_{j\mathrm{t}}}{\mathrm{d}\xi} + \frac{1}{r_j} \boldsymbol{b}_\mathrm{t} + \frac{\mathrm{d}}{\mathrm{d}\xi} \hat{\boldsymbol{n}} \times \boldsymbol{b}_\mathrm{t} \right) \frac{\Omega_j}{V}. \tag{4.31}$$

Secondly, one can rewrite the mutually driven and coupled-wave equations for the longitudinal electric field and transverse magnetic field in the concise forms

$$\mathscr{D}_E\, e_\mathrm{n} = \sum_j \frac{1}{\lambda_j^2} e_{j\mathrm{n}}, \tag{4.32}$$

$$\mathscr{D}_B\, \boldsymbol{b}_\mathrm{t} = \sum_j \frac{1}{\ell_j^2} \left(-\frac{V}{V_{j\mathrm{n}}} \boldsymbol{v}_{j\mathrm{t}} + \frac{V^2}{c_j^2 - V_{j\mathrm{n}}^2} (e_\mathrm{n} + e_{j\mathrm{n}}) (\hat{\boldsymbol{n}} \times \boldsymbol{v}_{j\mathrm{t}}) \right). \tag{4.33}$$

We recall again that up to this point of our algebraic derivations, no linearization has been made, and the density variations were entirely accounted for. Equation (4.29) or (4.30) permits to calculate the density of species j completely through the line integral over the electric fields that appears in the exponential Boltzmann factor in (4.30). So the density is a functional of the electric potentials. However, this dependence of n_j on ξ may now without loss of essential physics be neglected, if the density fluctuations can be assumed to remain small [i.e. we do not want to consider MHD shocks (Goossens, 2003), or electrostatic shocks and double layers here]. Thus, all scales and parameters, such as $V_{j\mathrm{n}}, c_j, \omega_j, \lambda_j, \ell_j$ and r_j, which have non-vanishing mean values, will, from here on, be calculated by use of the background number density $\bar{n}_j$, as well as the conditions for quasi-neutrality and zero longitudinal current and the centre-of-momentum condition. All compressive effects are described by the longitudinal electric field e_n in this approximation. Consequently, the nonlinear equations (4.31), (4.32) and (4.33) form a closed set, which yet will generally require a numerical treatment to obtain solutions. This is a task beyond the scope of this analytical paper.

Note that in each of these equations, the spatial variations are determined by the natural scales of the dynamics of the involved field variables, i.e. by the longitudinal scale r_j for the transverse motions of the particles, their Debye lengths λ_j for the charge fluctuations, respectively, their skin depths ℓ_j for the magnetic field penetration into the plasma driven by the transverse currents. Differential motion of the species j might be important and is therefore included in its drift speed $V_{j\mathrm{n}}$. Its effect on the parametric instabilities has been studied in work addressing the modulational and decay instability of Alfvén waves by considering streaming of alpha particles in the solar wind (Hollweg et al., 1993). If there are no differential motions along the mean field in the background plasma, i.e. if for all j we have $U_{j\mathrm{n}} = 0$, then $V_{j\mathrm{n}} = -V$, and thus the longitudinal gyration scale simply becomes $r_j = -V/\Omega_j$, which by its definition is not for each species a positive definite quantity as the gyrofrequency carries the sign of the charge of the species considered. The factor in front of the electric field term in the above equations (4.31) and (4.33) thus changes in the drift-free case to $1/(1 - (c_j/V)^2)$, which becomes $1/(1 - \beta_j)$ for $V = V_\mathrm{A}$, with the species plasma beta being defined as $\beta_j = (c_j/V_\mathrm{A})^2$. For the parallel propagation considered here, only this factor contains the thermal speed, and therefore this factor simply becomes unity for a cold multi-species plasma without drifts.

4.4. Electric field fluctuations driven by an elliptically polarized Alfvén wave

In this section, we consider the electrostatic waves, which can be generated by an elliptically polarized Alfvén wave, which for the sake of simplicity we assume to be given. We shall then study the effect this wave has in generating compressive fluctuations driven by the spatial variation of the kinetic energy of the particles moving coherently in the same wave magnetic field. The starting point is (4.32), which can with the help of (4.27) be written as a driven oscillator for the longitudinal electric field:

$$\mathscr{D}_E\, e_{\mathrm{n}} = -\sum_j \frac{V}{2\Omega_j}\frac{1}{\lambda_j^2}\frac{\mathrm{d}\boldsymbol{v}_{j\mathrm{t}}^2}{\mathrm{d}\xi}. \tag{4.34}$$

Before we write down the wave fields, let us define a proper coordinate system. For the right-handed orthogonal system, we chose the unit vectors $\hat{\boldsymbol{n}} = \boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$, $\boldsymbol{e}_1 = \boldsymbol{e}_2 \times \hat{\boldsymbol{n}}$, and $\boldsymbol{e}_2 = \hat{\boldsymbol{n}} \times \boldsymbol{e}_1$. The wave may have a wave vector k, and its normalized (dimensionless) magnetic field reads

$$\boldsymbol{b}_{\mathrm{t}} = b_1\boldsymbol{e}_1 \cos(k\xi) + b_2\boldsymbol{e}_2 \sin(k\xi). \tag{4.35}$$

Similarly, the related flow velocity of species j is given by

$$\boldsymbol{v}_{j\mathrm{t}} = v_{j1}\boldsymbol{e}_1 \cos(k\xi) + v_{j2}\boldsymbol{e}_2 \sin(k\xi). \tag{4.36}$$

The associated electric field according to (4.34) reads

$$e_{j\mathrm{n}} = \frac{kV}{\Omega_j}\cos(k\xi)\sin(k\xi)\left(v_{j1}^2 - v_{j2}^2\right). \tag{4.37}$$

On the other hand, using the original definition (3.9) of this field, we obtain the result

$$e_{j\mathrm{n}} = \cos(k\xi)\sin(k\xi)(b_2 v_{j1} - b_1 v_{j2}), \tag{4.38}$$

which by comparison of the last two equations, determines the particle velocity components as $v_{j1,2} = b_{2,1}\,\Omega_j/(kV)$. Finally, we have

$$e_{j\mathrm{n}} = \frac{\Omega_j}{2kV}\sin(2k\xi)\left(b_1^2 - b_2^2\right), \tag{4.39}$$

which can be inserted in (4.32). We obtain the forced oscillator equation

$$\frac{\mathrm{d}^2 e_{\mathrm{n}}}{\mathrm{d}\xi^2} + q^2 e_{\mathrm{n}} = \varepsilon\frac{\sin(2k\xi)}{2k}. \tag{4.40}$$

We recall that the Debye wave length λ_{D} was defined in (3.5). We used as an abbreviation the complex wave vector $q = i/\lambda_{\mathrm{D}}$, and further introduced the forcing amplitude as

$$\varepsilon = \sum_j \frac{1}{\lambda_j^2}\frac{\Omega_j}{V}(b_1^2 - b_2^2), \tag{4.41}$$

which vanishes for a circularly polarized wave with $b_1 = b_2$, but is nonzero otherwise. The solution of (4.40) can be obtained by using the Green's function method and Fourier transformation. The convolution integral of the forcing term with the

Green's function then yields the solution depending upon ξ in the form of another convolution integral, which can be calculated analytically with the (in q symmetric) result

$$e_{\rm n}(\xi) = \frac{\varepsilon}{4kq}\left(\frac{\sin(2k\xi)+\sin(q\xi)}{2k+q} - \frac{\sin(2k\xi)-\sin(q\xi)}{2k-q}\right). \tag{4.42}$$

Equation (4.42) solves the original equation (4.40) that can easily be shown by straightforward differentiation. We recall that by definition the square of the wave vector $q = q(V)$ is given by the right-hand side of the electrostatic dispersion relation (4.20). Therefore, q is a real number for an appropriate choice of V. The solution is then related naturally with the electrostatic eigenmodes, i.e. the sound, ion acoustic, and Langmuir waves, which we already discussed in a previous section. The overall solution (4.42) apparently describes forced compressive (charge) waves, occurring as electrostatic eigenmodes, and a superposed electric wave at the second harmonic of the transverse Alfvén pump wave, the anisotropy (due to its elliptic polarization) of which determines the amplitude of these driven longitudinal electric field oscillations. L'Hôpital's rule allows one to determine the behaviour in the resonant cases. For the resonances ($q \to \pm 2k$), we find

$$e_{\rm n}(\xi) = -\frac{\varepsilon}{2q^2}\left[\xi\cos(q\xi) - \frac{\sin(q\xi)}{q}\right], \tag{4.43}$$

which corresponds to the amplitude of the compressible oscillation growing or decaying with ξ, i.e. an instability in space.

5. Discussion and conclusions

Starting from the multi-fluid equations of a warm plasma, we have derived and investigated the coupled wave equations for the particles' gyromotions about the mean field and for the transverse magnetic field and longitudinal electric field. It is a natural outcome of the electromotive forces arising from compressible Alfvén-cyclotron waves and can be derived from a potential that is just the kinetic energy associated with the gyromotion in the electromagnetic wave. Electric waves are thus excited, which can react back on the pump wave by nonlinear effects through terms in its own wave equation that contains the electric field explicitly. Known limiting cases are reproduced, such as the standard linear electric waves like the ion-acoustic or Langmuir waves of course, and for the transverse magnetic field the usual two branches of Alfvén-cyclotron and magnetosonic-whistler waves in case of a two-component electron-proton plasma, or many similar related branches in the case of a multi-ion plasma. The main result of this paper is the closed set of second-order wave equations (4.31), (4.32) and (4.33), from solutions of which the transverse electric field and charge densities of each species can be derived as auxiliary quantities. To study these wave equations in more detail and to find their nonlinear solutions is left as a future task, which will presumably require a numerical treatment.

The structure of our equations already permits to derive some qualitative conclusions and to treat some simple applications (like the effect of elliptical polarization) analytically. Further study is certainly required to corroborate them quantitatively. Apparently, the weakly compressible large-amplitude Alfvén-cyclotron waves can drive electric fluctuations, essentially of the ion-acoustic type, along the mean field, and thus will naturally produce an electric field that can accelerate particles and will lead to heating via Landau damping in a kinetic Vlasov description. By excitation of acoustic waves, the amplitude of the driver wave will be diminished until a dynamic wave–wave equilibrium is reached. Similar processes are clearly found in the direct numerical simulations (Araneda et al., 2008, 2009; Valentini et al., 2008; Valentini and Veltri, 2009). The third-order coupling terms in (4.31) and (4.33) correspond to such three-wave processes in Fourier space, and therefore will lead to cascading of spectral energy and broadening of the original spectrum of the pump wave, which need not be monochromatic. This way a new path towards micro- and macro-turbulence could be opened, and a non-MHD cascade is rendered possible by these compressive Alfvén-cyclotron–acoustic wave interactions.

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